1 Number

Sets, exponents, surds, units and errors: the toolkit for working accurately with any number.

Key ideas

- **Number sets:** natural $\mathbb{N} \subset$ integers $\mathbb{Z} \subset$ rational $\mathbb{Q} \subset$ real $\mathbb{R}$. Irrational numbers (π, $\sqrt{2}$) are real but not rational.
- **Exponent laws:** $a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $(a^m)^n = a^{mn}$, $a^0 = 1$, $a^{-n} = 1/a^n$
- **Scientific notation:** $a \times 10^n$ with $1 \leq a < 10$. Round to decimal places (d.p.) or significant figures (s.f.).
- **Surds:** $\sqrt{(ab)} = \sqrt{a} \times \sqrt{b}$. Simplify by taking out square factors: $\sqrt{50} = 5\sqrt{2}$.
- **Percentage error** = |estimate − exact| ÷ exact × 100%

How to do it

1. **Simplify a surd:** find the largest square factor. $\sqrt{72} = \sqrt{(36 \times 2)} = 6\sqrt{2}$
2. **Add surds:** only like surds combine. $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$
3. **Convert units:** × going to a smaller unit, ÷ going to a bigger one. For area square the factor, for volume cube it.
4. **Percentage error:** find the absolute error, divide by the exact value, then × 100.

Worked example

Simplify $\sqrt{12} + \sqrt{75}$

→ $\sqrt{12} = \sqrt{(4 \times 3)} = 2\sqrt{3}$

→ $\sqrt{75} = \sqrt{(25 \times 3)} = 5\sqrt{3}$

→ $2\sqrt{3} + 5\sqrt{3} = \mathbf{7\sqrt{3}}$

Watch out!

- 1 m² = 10 000 cm², not 100 cm².
- 0.0760 to 3 s.f. keeps its last zero.
- $\sqrt{a} + \sqrt{b} \neq \sqrt{(a + b)}$: $\sqrt{9} + \sqrt{16} = 7$, not 5.

Kong's tips

- Write units and rounding with your answer: 352 m³ (3 s.f.).
- Estimate first (round to 1 s.f.) to check your calculator answer.

Level 7–8 challenge

Unfamiliar, multi-step problems: show every step and explain your answer.

Q1 A tank is 2.4 m long, 1.5 m wide and 80 cm deep. How many litres does it hold (1 m³ = 1000 L)? It fills at 12 L per minute. How long does it take to fill?

Check your answer: V = 2.4 × 1.5 × 0.8 = 2.88 m³ = 2880 L. 2880 ÷ 12 = 240 minutes = 4 hours.

Q2 A square has an area of 50 cm². Find its exact side length and perimeter, giving both as simplified surds.

Check your answer: Side = $\sqrt{50} = \sqrt{(25 \times 2)} = 5\sqrt{2}$ cm. Perimeter = $4 \times 5\sqrt{2} = 20\sqrt{2}$ cm.

2 ♠

Coordinate geometry

Use coordinates to find lengths, midpoints and gradients, write equations of lines and solve two equations together.

Key ideas

- **Distance:** $d = \sqrt{((x_2 - x_1)^2 + (y_2 - y_1)^2)}$
- **Midpoint:** $M = ((x_1 + x_2)/2, (y_1 + y_2)/2)$
- **Gradient:** $m = (y_2 - y_1) \div (x_2 - x_1)$. **Parallel:** same m. **Perpendicular:** $m_1 \times m_2 = -1$
- **Equation of a line:** $y = mx + c$ (m = gradient, c = y-intercept)
- **Simultaneous equations:** two equations, two unknowns. The solution is where the two lines cross.

How to do it

1. **Line through two points:** find m, then put one point into $y = mx + c$ to find c.
2. **Prove a shape:** use distances for equal sides, gradients for parallel or perpendicular sides.
3. **Substitution:** rearrange one equation for x or y, put it into the other.
4. **Elimination:** make one variable's coefficients match, then add or subtract the equations.

Worked example

Find the line through (1, 6) and (3, 10)

→ $m = (10 - 6) \div (3 - 1) = 2$

→ $6 = 2(1) + c$, so $c = 4$

→ **$y = 2x + 4$**

Watch out!

- Subtract coordinates in the same order, top and bottom.
- A perpendicular gradient is flipped AND negative: $2 \rightarrow -\frac{1}{2}$.
- Check your solution in BOTH equations.

Kong's tips

- Sketch the points first: it shows if the gradient should be positive or negative.
- Word problems: define your letters before writing equations.

Level 7–8 challenge

Unfamiliar, multi-step problems: show every step and explain your answer.

Q1 A(1, 2), B(4, 6) and C(8, 3) form a triangle. Show that it is right-angled and isosceles, and find its area.

Check your answer: $AB = \sqrt{(3^2 + 4^2)} = 5$ and $BC = \sqrt{(4^2 + 3^2)} = 5$. Gradients 4/3 and −3/4 multiply to −1, so the right angle is at B. Area = $\frac{1}{2} \times 5 \times 5 = 12.5$ units².

Q2 Phone plan A costs \$30 plus \$0.05 per minute. Plan B costs \$10 plus \$0.15 per minute. When do they cost the same? Which is cheaper for 300 minutes?

Check your answer: $30 + 0.05m = 10 + 0.15m \rightarrow 20 = 0.1m \rightarrow m = 200$ minutes (\$40 each). At 300 minutes: A = \$45, B = \$55, so A is cheaper.

3 ♣

Functions & quadratics

A function gives every input exactly one output. Quadratics make U-shaped graphs called parabolas.

Key ideas

- **Function:** every input (domain) has exactly one output (range). On a graph, use the vertical line test.
- **Factorising:** $x^2 + bx + c = (x + p)(x + q)$, where $p + q = b$ and $pq = c$
- **Difference of two squares:** $a^2 - b^2 = (a + b)(a - b)$
- **Vertex form:** $y = a(x - h)^2 + k$ has vertex (h, k). $a > 0$ → concave up, $a < 0$ → concave down.
- **Quadratic formula:** $x = (-b \pm \sqrt{(b^2 - 4ac)}) \div 2a$. If $b^2 - 4ac < 0$, there are no real solutions.

Worked example

Solve $x^2 - x - 6 = 0$

→ Factors of −6 that add to −1: −3 and 2

→ $(x - 3)(x + 2) = 0$

→ **$x = 3$ or $x = -2$**

Watch out!

- Rearrange to = 0 before factorising.
- $(x + 3)^2 \neq x^2 + 9$. It is $x^2 + 6x + 9$.
- In the formula, −b is positive when b is negative.

How to do it

1. **Solve by factorising:** make it = 0, factorise, set each bracket to 0.
2. **Complete the square:** $x^2 + bx + c = (x + b/2)^2 - (b/2)^2 + c$
3. **Sketch:** find the y-intercept ($x = 0$), the roots ($y = 0$) and the vertex, then draw a smooth curve.
4. **Won't factorise?** Use the quadratic formula.

Kong's tips

- Check each root by putting it back into the equation.
- The vertex is halfway between the two roots.

Level 7–8 challenge

Unfamiliar, multi-step problems: show every step and explain your answer.

Q1 A ball's height is $h = -5t^2 + 20t + 25$ metres after t seconds. Find its maximum height and when it hits the ground.

Check your answer: Vertex at $t = 20 \div 10 = 2$ s, so $h = -20 + 40 + 25 = 45$ m. Ground: $t^2 - 4t - 5 = 0$ → $(t - 5)(t + 1) = 0$, so $t = 5$ s.

Q2 A garden's length is 3 m more than its width, and its area is 40 m². Find its dimensions.

Check your answer: $w(w + 3) = 40$ → $w^2 + 3w - 40 = 0$ → $(w + 8)(w - 5) = 0$. Width can't be negative, so 5 m by 8 m.

Sets & probability

Venn diagrams sort things into sets. Probability tells you how likely each outcome is.

Key ideas

- **A ∩ B** (intersection): in both. **A ∪ B** (union): in either. **A'**: not in A. **U**: everything.
- **P(A)** = favourable outcomes ÷ total outcomes, and P(A') = 1 − P(A)
- **Independent events:** P(A and B) = P(A) × P(B)
- **Mutually exclusive:** can't both happen, so P(A or B) = P(A) + P(B)
- **Without replacement:** the probabilities on the second branches change.

How to do it

1. **Venn diagram:** fill the overlap first, then the rest of each circle, then the outside.
2. **Tree diagram:** multiply ALONG the branches, add between different paths.
3. **"At least one":** work out 1 − P(none).
4. **Two-way table:** for "given that", use only that row or column total.

Worked example

5 red and 3 blue counters. Take 2 without replacement. P(both red)?

→ First: 5/8. Second: 4/7 (one red gone)

→ 5/8 × 4/7 = 20/56

→ = **5/14**

Watch out!

- Without replacement, the total drops by one on the second pick.
- Each set of branches must add up to 1.
- Don't count the overlap twice: n(A ∪ B) = n(A) + n(B) − n(A ∩ B).

Kong's tips

- Check that all the numbers in your Venn diagram add up to the total.
- Keep probabilities as fractions until the end.

Level 7–8 challenge

Unfamiliar, multi-step problems: show every step and explain your answer.

Q1 In a class of 30, 18 play football, 12 play tennis and 5 play neither. How many play both? What is the probability that a student plays only tennis?

Check your answer: 25 play at least one, so 18 + 12 − both = 25, giving both = 5. Only tennis = 12 − 5 = 7, so P = 7/30.

Q2 A fair coin is tossed three times. Find P(exactly two heads) and P(at least one tail).

Check your answer: 8 outcomes. Exactly two heads: HHT, HTH, THH = 3/8. At least one tail = 1 − P(HHH) = 1 − 1/8 = 7/8.

5 ♦

3D shapes & transformations

Find the surface area and volume of solids, and move shapes with translations, reflections, rotations and enlargements.

Key ideas

- **Cylinder:** $V = \pi r^2 h$, $SA = 2\pi r^2 + 2\pi rh$
- **Cone:** $V = \frac{1}{3}\pi r^2 h$, $SA = \pi r^2 + \pi rl$ (l = slant height). **Pyramid:** $V = \frac{1}{3} \times$ base area $\times h$
- **Sphere:** $V = 4/3\ \pi r^3$, $SA = 4\pi r^2$
- **Translation** by vector (a, b): $(x, y) \rightarrow (x + a, y + b)$
- **Reflection** needs a mirror line, **rotation** an angle, direction and centre, **enlargement** a scale factor and centre.

How to do it

1. **Solids:** name the shape, write the formula, then substitute.
2. **Cone slant height:** $l = \sqrt{(r^2 + h^2)}$
3. **Describe a transformation fully:** its type plus every detail it needs.
4. **Find a mirror line:** it is halfway between each point and its image.

Worked example

Cone: r = 3 cm, h = 4 cm

→ $l = \sqrt{(9 + 16)} = 5$ cm

→ $V = \frac{1}{3}\pi(9)(4) = 12\pi \approx$ **37.7 cm³**

→ $SA = 9\pi + 15\pi = 24\pi \approx$ **75.4 cm²**

Watch out!

- Volume uses the perpendicular height; a cone's surface uses the slant height.
- A rotation needs a direction AND a centre.
- Scale factor k multiplies area by k^2 and volume by k^3.

Kong's tips

- Keep π in your answer until the last step.
- Use tracing paper to check rotations.

Level 7–8 challenge

Unfamiliar, multi-step problems: show every step and explain your answer.

Q1 A metal ball of radius 3 cm is melted and recast into a cylinder of radius 2 cm. How tall is the cylinder?

Check your answer: Ball: $V = 4/3\ \pi(3^3) = 36\pi$ cm³. Cylinder: $\pi(2^2)h = 4\pi h$. $4\pi h = 36\pi$, so $h = 9$ cm.

Q2 Triangle A(1, 1), B(4, 1), C(1, 3) is enlarged by scale factor 2 about the origin. Find the new vertices. How many times bigger is the area?

Check your answer: A'(2, 2), B'(8, 2), C'(2, 6). Area goes from 3 to 12 units²: 4 times bigger ($2^2 = 4$).

6 ♠

Statistics

Summarise data with averages and spread, and show it with box plots, cumulative frequency graphs and histograms.

Key ideas

- **Averages:** mean, median, mode. **Spread:** range, and IQR = Q3 − Q1
- **Outlier:** more than 1.5 × IQR below Q1 or above Q3
- **Grouped mean** (an estimate): ∑(midpoint × frequency) ÷ ∑frequency
- **Cumulative frequency:** plot running totals at each class's upper boundary. Median at n/2, Q1 at n/4, Q3 at 3n/4.
- **Histogram:** bars touch (continuous data). Describe the shape: symmetrical, skewed, unimodal, bimodal.

How to do it

1. **Estimate the mean:** midpoint × frequency for each class, add them up, divide by the total frequency.
2. **Box plot:** draw the five-number summary (min, Q1, median, Q3, max) on a scale.
3. **Compare two data sets:** one average AND one measure of spread, in context.

Worked example

Estimate the mean: 0–10 (8), 10–20 (12), 20–30 (16), 30–40 (11), 40–50 (3)

→ Midpoints: 5, 15, 25, 35, 45

→ ∑(mid × f) = 1140 and ∑f = 50

→ Mean ≈ 1140 ÷ 50 = **22.8**

Watch out!

- A grouped mean is only an estimate.
- Plot cumulative frequency at the END of each class.
- "Compare" needs a sentence in context, not just numbers.

Kong's tips

- Outliers or skewed data? Use the median, not the mean.
- Read values off a curve with a ruler for accuracy.

Level 7–8 challenge

Unfamiliar, multi-step problems: show every step and explain your answer.

Q1 Data: 12, 15, 17, 18, 20, 22, 25, 61. Find the quartiles and IQR, then decide whether 61 is an outlier. Justify.

Check your answer: Median 19, Q1 = 16, Q3 = 23.5, IQR = 7.5. Upper limit = 23.5 + 1.5 × 7.5 = 34.75. 61 > 34.75, so it is an outlier.

Q2 Class A: median 65, IQR 20. Class B: median 72, IQR 8. Compare the two classes' test results.

Check your answer: Class B scored higher on average (median 72 vs 65) and more consistently (IQR 8 vs 20).

7 ♣

Trigonometry & circles

Use trigonometry in right triangles, measure arcs and sectors, and prove angle facts with circle theorems.

Key ideas

- **SOH CAH TOA:** $\sin\theta = O/H$, $\cos\theta = A/H$, $\tan\theta = O/A$
- **Exact values:** $\sin 30° = \cos 60° = \frac{1}{2}$, $\tan 45° = 1$, $\sin 60° = \cos 30° = \sqrt{3}/2$
- **Arc length** $= \theta/360 \times 2\pi r$. **Sector area** $= \theta/360 \times \pi r^2$
- **Circle theorems:** angle in a semicircle = 90° · angle at centre = 2 × angle at circumference · angles in the same segment are equal · opposite angles in a cyclic quadrilateral add to 180°
- **Intersecting chords:** AX × XB = CX × XD

Worked example

From 40 m away, a tower's top is at 35° elevation. Height?

→ opp and adj → TOA: $\tan 35° = h/40$

→ $h = 40 \times \tan 35°$

→ $h \approx$ **28.0 m**

Watch out!

- Calculator in DEG mode.
- Sector perimeter = arc + 2 radii.
- No reason, no marks: name the theorem.

How to do it

1. **Trig:** label the sides from the angle, choose the ratio, solve. Use $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ for angles.
2. **Elevation and depression:** draw the horizontal first, then the right triangle.
3. **Bearings:** clockwise from north, always 3 figures (073°).
4. **Circle proofs:** give a reason for every step.

Kong's tips

- Mark the centre and every radius: radii make isosceles triangles.
- Check the hypotenuse is the longest side.

☆ Level 7–8 challenge

Unfamiliar, multi-step problems: show every step and explain your answer.

Q1 A ship sails 8 km due east (090°), then 6 km due south (180°). How far is it from its start, and on what bearing must it sail to go straight back?

Check your answer: Distance $= \sqrt{(8^2 + 6^2)} = 10$ km. Back is 6 km north and 8 km west: $\tan^{-1}(8/6) \approx 53.1°$ west of north, so bearing ≈ 306.9°.

Q2 O is the centre of a circle and ∠AOC = 130°. B is on the major arc and D on the minor arc. Find ∠ABC and ∠ADC, giving reasons.

Check your answer: ∠ABC = 65° (angle at the centre is twice the angle at the circumference). ∠ADC = 180° − 65° = 115° (opposite angles in a cyclic quadrilateral).

8 ♥

Sequences & variation

Find rules for number patterns, and describe how one quantity changes when another one does.

Key ideas

- **Linear sequence:** common difference d, so $u_n = dn + (u_1 - d)$
- **Quadratic sequence:** second differences are constant. $u_n = an^2 + bn + c$, where $2a$ = second difference
- **Direct variation:** $y = kx$ (or kx^2, kx^3). **Inverse variation:** $y = k/x$ (or k/x^2)
- **Find k** by substituting one known pair of values.
- **Change the subject:** do the same inverse operation to both sides.

Worked example

Find the nth term of 7, 11, 15, 19, ...

→ d = 4, and 4n gives 4, 8, 12, ...

→ Each term is 3 more, so $\mathbf{u_n = 4n + 3}$

→ $u_{20} = 4(20) + 3 = \mathbf{83}$

Watch out!

- The nth term of 7, 11, 15 is 4n + 3, not n + 4.
- Inverse is not negative: $y = k/x$, not $y = -kx$.
- A square root gives ± answers.

How to do it

1. **nth term of a linear sequence:** find d, compare dn with the sequence, then add or subtract the difference.
2. **Variation problems:** write the equation with k, find k, rewrite it with k, then answer.
3. **Rearranging:** undo + and − first, then × and ÷, then powers and roots.

Kong's tips

- Test your nth term with n = 1 and n = 2.
- In $y = kx^2$, doubling x makes y four times bigger.

☆ Level 7–8 challenge

Unfamiliar, multi-step problems: show every step and explain your answer.

Q1 A car's stopping distance d varies directly as the square of its speed v. At 40 km/h, d = 12 m. Find d at 100 km/h.

Check your answer: $d = kv^2$. $12 = k \times 40^2$, so $k = 0.0075$. $d = 0.0075 \times 100^2 = 75$ m.

Q2 Is 302 a term of the sequence 5, 12, 19, 26, ...? Explain how you know.

Check your answer: $u_n = 7n - 2$. $7n - 2 = 302$ gives $n = 43.4...$, which is not a whole number, so 302 is not a term.

9 ♦

Sampling & bivariate data

Choose a fair sample from a population, and look for relationships between two variables.

Key ideas

- **Population** = everyone. **Sample** = the part you survey. A **sampling frame** lists everyone you could pick.
- **Random:** everyone equally likely. **Systematic:** every kth member from a random start.
- **Stratified:** sample each group in proportion to its size. **Quota:** fill a set number per group (not random).
- **Correlation:** positive, negative or none; strong or weak. It does NOT prove cause.
- **Line of best fit:** goes through the mean point ($\bar{x}$, $\bar{y}$). Predict inside the data range only.

Worked example

900 staff: 300 in sales, 600 in production. Stratified sample of 60.

→ Sales: 300/900 × 60 = **20**

→ Production: 600/900 × 60 = **40**

→ Check: 20 + 40 = 60

Watch out!

- Asking friends or volunteers gives a biased sample.
- Correlation is not causation.
- Extrapolating beyond the data is unreliable.

How to do it

1. **Stratified sample size:** group size ÷ population × sample size.
2. **Describe correlation:** direction + strength + what it means in context.
3. **Line of best fit:** plot ($\bar{x}$, $\bar{y}$), then draw a line through it that follows the trend.
4. **Predict** by reading from the line or using its equation.

Kong's tips

- Explain WHY a sample might be biased, not just that it is.
- Draw the line of best fit with points on both sides of it.

Level 7–8 challenge

Unfamiliar, multi-step problems: show every step and explain your answer.

Q1 A school has 420 students: 140 in Year 9, 160 in Year 10 and 120 in Year 11. Choose a stratified sample of 42. Why is this better than asking the first 42 students in the canteen?

Check your answer: Year 9: 14, Year 10: 16, Year 11: 12. The canteen sample isn't random and may over-represent some years or groups.

Q2 The line of best fit for hours studied (x) against test score (y) is y = 6x + 40, from data for 1–8 hours. Predict the score for 5 hours. Should you use it for 15 hours?

Check your answer: y = 6(5) + 40 = 70. No: 15 hours is outside the data, and it predicts 130, which is impossible on a test out of 100.